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Monomial cubature rules since "Stroud": a compilation

Ronald Cools
Department of Computer Science, Katholieke Universiteit Leuven, Belgium

Philip Rabinowitz
Department of Applied Mathematics & Computer Science, The Weizmann Institute of Science, Rehovot, Israel

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Abstract

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A bibliography of references to cubature rules which have appeared since the publication of Stroud's book (1971) is presented. The standard regions that are treated in this paper are the n-cube, the n-simplex, the n-sphere and the entire space.

Keywords: Cubature rule; multiple integration rule.

1. Introduction

Over two decades ago, Stroud published his encyclopedic work on multiple numerical integration, *Approximate Calculation of Multiple Integrals* [98]. In this book, Stroud presented a rather complete summary of the theoretical and practical aspects of multiple numerical integration, a comprehensive bibliography and a listing of almost all multiple integration or cubature rules (CRs) for a variety of regions. This book has proved very useful to all workers in the field and very few papers concerned with multiple numerical integration omit it from their list of references.

Now, the study of multiple numerical integration has continued unabatedly during the past two decades. Much of what was done until 1984 is summarized in [24], which contains an

Correspondence to: Dr. R. Cools, Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Leuven, Belgium. e-mail: ronald.cools@cs.kuleuven.ac.be.

extensive bibliography. Some later references can be found in [9,10,37,60,78]. This paper is concerned with continuing the work of Stroud in one specific area, namely the compilation of all so-called monomial CRs which have appeared since the publication of [98] plus some CRs which appeared earlier but were not included in [98] for some reason. We should qualify the word "all" in several ways. First, we have restricted ourselves to four regions so that several regions treated in [98] are not included in our compilation, such as the surface of the sphere, the spherical shell, the hexagon, the octahedron, etc. Second, we have ignored the Russian and other nonwestern literature, except when it has appeared in translation. The reader who has access to Russian publications is referred to [81] for information about this literature. Third, we have not included every CR we have seen, omitting CRs which did not satisfy certain reasonable criteria which we give below. Finally, we have surely overlooked some CRs which have appeared in the accessible literature and, a fortiori, CRs which have appeared in theses, technical reports, research papers, etc. If the reader has any information about such CRs, he is requested to inform the authors.

We remark that this is not the first compilation that has been made. Previous compilations have appeared in [10,72] and, indeed, they have been very useful in the preparation of the present work. However, they were limited in scope and not as comprehensive as the present compilation and certainly not as up-to-date. In the next section, we give the necessary background and description of the tables including some guidelines for deciding which CRs were included and which sources are given for a CR when it has several sources.

2. Background material and description of the tables

We consider CRs of precision or degree d of the form

$$\int_{B_n} w(\boldsymbol{x}) f(\boldsymbol{x})\,\mathrm{d}V = I_N f + R_N f, \tag{1}$$

where B_n is a region in an n-dimensional Euclidean space E_n, $\boldsymbol{x} := (x_1, \ldots, x_n)$ is a point in E_n, $\mathrm{d}V$ is an element of volume in E_n, w is a weight function,

$$I_N f := \sum_{i=1}^{N} w_i f(\boldsymbol{x}_i), \quad w_i \neq 0,\ i = 1, \ldots, N, \tag{2}$$

is a CR and $R_N f$ is the remainder which has the property that $R_N f = 0$ if f is a linear combination of monomials of the form

$$\prod_{j=1}^{n} x_j^{i_j},$$

with $\sum_{j=1}^{n} i_j \leqslant d$ and $R_N f \neq 0$ for some such monomial with $\sum_{j=1}^{n} i_j = d + 1$.

The regions B_n which we include in our compilation are the three bounded regions C_n, the hypercube, S_n, the hypersphere, and T_n, the simplex, and the entire space E_n. Associated with the bounded regions is the weight function $w(\boldsymbol{x}) \equiv 1$ while with E_n we associate two weight functions, $\exp(-r)$ and $\exp(-r^2)$ where $r^2 := \sum_{j=1}^{n} x_j^2$. As in [98], E_n with these two weight functions is denoted by E_n^r and $E_n^{r^2}$, respectively. The definitions of the bounded regions are given by

$$\begin{aligned} C_n&: \quad -1 \leqslant x_j \leqslant 1, \ j=1,\dots,n, \\ S_n&: \quad r^2 \leqslant 1, \\ T_n&: \quad \sum_{j=1}^{n} x_j \leqslant 1, \ x_j \geqslant 0, \ j=1,\dots,n. \end{aligned}$$

In our compilation, we give individual tables with information about CRs for the following regions of particular dimension: C_n, T_n ($n=2, 3, 4$), S_n, E_n^r ($n=2, 3$), $E_n^{r^2}$ ($n=2, 3, 5$) and one additional table for C_n, S_n, T_n and $E_n^{r^2}$ of general dimension.

In most cases, we also add a separate table of embedded sequences of CRs. These sequences are either fully or partially embedded. In a fully embedded sequence, each CR of higher degree includes among its integration points each point of all the CRs of lower degree in the sequence. In a partially embedded sequence, some points in a CR of lower degree do not appear among the points of a CR of higher degree. Although, in theory, any sequence of CRs can be considered to be a partially embedded sequence, in practice we require that the number of points not appearing in the CR of higher degree be "small". If we relax the condition in (2) that all weights $w_i \neq 0$, then we can look upon a partially embedded sequence as a fully embedded sequence in which some weights vanish. In our tables, we take this approach and indicate partial embedding by the letter P in the column labeled N. We remark that sometimes CRs appearing in an embedded sequence are also included among the individual CRs when they are more efficient than some of the CRs included in the tables, where we define one CR over a region B_n to be more efficient than another CR over the same region if both are of the same degree and the first CR uses fewer points than the second.

As a general rule we do not include CRs of degree $2m-1$ which use more than m^n points since the (Cartesian, spherical or conical) product Gaussian rule of degree $2m-1$ requires m^n points and is usually superior to any other CR of the same degree. This also holds true for the top of an embedded sequence, since there is a very good way to generate an embedded sequence starting with a product Gaussian rule and generating CRs of lower degree in an optimal manner [21]. Exceptions to this rule are CRs with properties that some users may desire, such as equal weights, points on the boundary or more symmetry.

In our tables, we give the following information about CRs and embedded sequences in addition to the degree d and the corresponding number of integration points N, namely the quality and the references. The quality of a CR is given by one or two letters which use a slight extension of the notation introduced in [68]. The first letter gives information about the weights w_i. If all w_i are positive, we denote this by the letter P, unless they are all equal, in which case we use the letter E. If some w_i are negative, we indicate this by the letter N. Occasionally, only the points of a CR are published and not the weights, since these can be evaluated quite easily by solving a system of linear equations. In this case, we replace the first letter with a question mark. The second letter only appears for bounded regions B_n and gives information about the location of the integration points $\boldsymbol{x}_i$. If all $\boldsymbol{x}_i \in B_n$, we indicate this by the letter I, unless some of the $\boldsymbol{x}_i$ lie on the boundary of B_n, in which case we use the letter B. If some $x_i \notin B_n$, we write O. In the case of an embedded sequence, consisting of p CRs, p or $p+1$ letters are used depending on whether B_n is the entire space or not. In the latter case, the last letter gives information about the integration points in the last CR of the sequence, namely the one of highest degree. In all cases, the first p letters give information about the weights of all the CRs in the sequence, starting with the CR of lowest degree. In the quality column, we also indicate the number of CRs which have the same values of d and N and the same quality. If there is a one-parameter family of CRs with the same characteristics, we indicate this by ∞.

We now give the guidelines for the entries in the references column which we have tried to keep as short as possible although we try to supply all useful references. As principal reference, we give the first appearance of the CR in a journal or book, even though it has previously appeared in an internal report or thesis. At times, this reference is supplemented by a second reference which corrects some deficiencies in the first reference or which derives the same CR in a different context. Another case where additional references are included occurs when a later paper contains all of the CRs in an earlier paper plus many additional ones. In this case, we wish to spare the user the trouble of consulting two references when all the information he requires appears in the later paper. Internal reports and theses are only mentioned if the CR did not appear in a journal or book. On inspecting the CRs in the tables, the reader may notice that in many cases, less efficient CRs were computed after the appearance of more efficient ones either in [98] or elsewhere. One possible explanation for this is that the less efficient CR was computed as an example to illustrate a particular theory for the construction of CRs. The reader may also notice that we have included some CRs from [98], even though our explicit aim was to present CRs which did not appear in [98]. One reason for this is that in many cases the CR in [98] is minimal and we wished the reader to have this information since we append an asterisk to all CRs known to contain the theoretically minimal number of points. Another reason is to avoid gaps in the tables as one proceeds from one value of d to another value $d_1 > d + 2$. Finally, a CR from [98] may not be minimal among all CRs of a given degree, but still it has some additional useful feature which is not present in other CRs. So, if [98] lists a formula that is PI and none of the known formulae with less nodes is PI, we include a reference to [98]. We do not give additional references to CRs that appeared in [98].

All the CRs in our tables are CRs for which either numbers or formulas are given in the references for the points $\boldsymbol{x}_i$ and, in most cases, for the weights w_i. Hence, we did not include the CRs in [55] which are given as functions of the moments, nor the CRs in [13,58] which are of considerable interest and which can be generated by the algorithm given there.

3. Cubature formulae for two-dimensional regions

3.1. Cubature formulae for the square C_2

Degree	N	Quality	References
3	4*	PI	[98]
	5	NI	[82]
4	6*	PI	[91]
		PI	[65]
		PI(2)	[101]
		PO	[65]
5	7*	PI	[98]
	8	PI	[38]
		PI	[76]
6	10*	PI	[92]
		PI	[91]
		PI(2)	[101]
		PO	[91]
7	12*	PI	[98]
		PO	[98]
		PI	[56]
		PI(∞)	[75]
		PI	[51]
		PI(4)	[65]
		PI(∞)	[65]
		PI(∞)	[92]
	13	PO	[39]
		?I(∞)	[92]
	14	PI	[39]
		PO	[39]
8	15*	PO	[79]
	16	PI(2)	[101]
9	17*	PI	[75]
	18	PI(2)	[88]
	20	PI	[39]
		PI	[11]
		PO	[30]
	21	NI	[39]
	23	PI	[39]
	24	PI	[39]
11	24*	PI	[18]
	25	NI	[87,50]
	26	NI	[87]
		NI	[50]
	28	NO	[69]
	30	PO	[56]
		NO	[80]
	33	NI	[82]
13	33	PI	[18]
	34	NO	[19]
	36	PO	[30]
		NO	[11]
	40	PI	[11]
		PO	[11]
	56	NI	[82]
15	44	NO	[98]
	45	NO	[30]
	48	NI	[70]
17	57	NO(2)	[11]
	60	PI	[30]
19	68	NO	[99]
	72	PO	[30]
21	85	NO	[79]
	88	PO	[30]
31	200	NO	[80]

Embedded cubature formulae

Degrees	N	Quality	References
3, 7	5, 17	NNI	[82]
5, 7	13, 17	NNI	[42]
	8, 16	PNI(2)	[12]
	12, 16	NPI	[35]
7, 9	12, 25	PPI	[12]
	13, 25	PPI	[12]
	13, 25	PNI	[12]
	17, 25	PPI	[35]
1, 3, 5, 7, 9	1, 5, 13, 21, 25	P^5I	[21]
1, 3, 5, 7, 9, 11, 13	See [43, p.586] with $n = 2$	?I	[43]
1, 3, 5, 7, 9, 11, 13, 15, 17	See [43, p.586] with $n = 2$	?O	[43,22]
9, 11	18, 36	PNI(2)	[12]
9, 15	41, 49 P	?PI(3)	[89]
11, 13	33, 65	NNI	[82]
11, 17	33, 97	NNI	[82]

3.2. Cubature formulae for the circle S_2

Degree	N	Quality	References
4	6*	PI	[98]
		?O	[92]
	10	EI	[32]
5	7*	PI	[98]
	8	PO	[11]
	12	EI	[32]
		EI(2)	[28]
6	10*	PO	[90,101]
		PI	[62]
	11	PO	[101]
	14	EI	[32]
7	12*	PI	[98]
	16	EI	[32]
	18	EI	[32]
	20	EI	[32]
8	16	PI	[101]
9	18*	PO	[88]
	19	PI	[74]
		PI	[51]
	20	PI(3)	[11]
		PO(2)	[11]
	21	PI	[11]

Degree	N	Quality	References
11	25	PO	[50]
	26	PI	[87]
		PI	[50]
13	34	PO	[19]
	35	PB	[19]
	36	PI	[15]
	41	PI	[46]
15	44	PI	[98]
17	61	PI	[98]
19	71	PO	[46]
	76	PI	[46,34]
21	90	PI	[46]
	99	PI	[46,13]
23	97	PO	[46]
	108	PI	[46,34]
25	127	PI	[46]
27	140	PI	[46,34]
31	172	PI	[46,34]

Embedded cubature formulae

Degrees	N	Quality	References
5, 7	8, 16	PNI(2)	[12]
17, 27	71, 183	PNI	[16]

3.3. Cubature formulae for the plane $E_2^{r^2}$

Degree	N	Quality	References	Degree	N	Quality	References	Degree	N	Quality	References
7	12*	P	[98]	13	34	P	[19]	21	90	P	[46]
					36	P	[15]		99	P	[46]
9	18*	P	[51]		41	P	[46]				
	19	P	[46]					23	97	P	[46]
		P	[51]	15	44	P	[98]		108	P	[46]
	20	P	[11]			P	[47]				
	21	P	[11]					25	127	P	[46]
				17	61	P	[46]				
11	25	P	[50]					27	140	P	[46]
	26	P	[87]	19	71	P(2)	[46]				
		P	[50]		76	P	[46]	31	172	P	[46]
	28	P	[46]								

Embedded cubature formulae

Degrees	N	Quality	References
$3, 5, \ldots, 2m+1$	See [43, p.586] with $n = 2$	(∞)	[22]
5, 7	8, 16	PN(2)	[12]
7, 9	12, 25	PN	[12]
11, 13	25, 49	PN	[12]
9, 15	19, 61	PP	[14]
9, 25	29, 141	PP	[14]
13, 25	43, 155	PP	[14]
17, 23, 39	71, 155, 393	PNN	[14]
17, 23, 43	71, 155, 449	PNN	[14]

3.4. Cubature formulae for the plane E_2^r

Degree	N	Quality	References	Degree	N	Quality	References	Degree	N	Quality	References
7	12*	P	[98]	13	34	P	[10]	21	90	P	[46]
		P	[51]		36	P	[15]		99	P	[46]
					41	P	[46]				
9	19	P	[46]					23	97	P	[46]
		P	[51]	15	44	P	[98]		108	P	[46]
	20	P	[11]			P	[47]				
	21	P	[11]					25	127	P	[46]
				17	61	P	[46]				
11	26	P	[50]					27	140	P	[46]
		P	[87]	19	71	P	[46]				
	28	P	[46]		76	P	[46]	31	172	P	[46]

Embedded cubature formulae

Degrees	N	Quality	References
5, 7	8, 16	PN	[12]
7, 9	13, 25	PN	[12]
9, 19	19, 85	PP	[14]
9, 23	19, 109	PN	[14]
9, 25	29, 141	PP	[14]
13, 19	41, 91	PN	[14]
17, 23	71, 155	PN	[14]
5, 9, 19	7, 25, 85	PPN	[14]
5, 9, 27	7, 25, 157	PPN	[14]

3.5. Cubature formulae for the triangle T_2

Degree	N	Quality	References
2	3*	PI	[98]
		PI(2)	[53]
		PB	[98]
		PB	[53]
	4	EI	[33,34]
		EI	[53]
		NB	[25]
3	4*	NI	[98]
		PI(2)	[53]
	5	EI	[33,34]
		PI	[53]
4	6*	PI	[23,29,64,68,73,97]
		?I	[92]
	7	PI	[23,64,73,97]
		PB	[53]
	8	EI	[33,34]
	9	PB	[68]
		NB	[25]
	10	NB	[68]
5	7*	PI	[98]
	9	PI	[23,64,73,97]
		PI	[53]
	10	PB	[68]
		NI	[75,44,25]
6	12	PI	[23,29,64,68,73,97]
		PI	[2]
	13	PB	[68]
		PB	[53]
	16	NB	[68]
7	12*	PI	[40]
	13	NI	[23,29,64,68,73,97]
	14	PI(2)	[38]
		NO	[45]
	15	PI	[64]
	16	PB	[68]
		PB	[53]

Degree	N	Quality	References
8	15*	PO	[17]
	16	PI	[68,29,64]
		NI	[68]
	21	NI	[68]
9	19	PI	[68,29,64]
	21	PB	[64]
	22	PB	[68]
10	22	PO	[17]
	25	PI(2)	[64]
	25	PI	[29]
11	27	PO	[68,29]
	28	PI	[68]
12	33	PI	[29]
13	36	NO	[8]
	37	PI	[29]
		PI	[8]
		PB	[8]
	40	PB	[8]
14	42	PI	[29]
15	48	PO	[29]
16	52	PO	[29]
17	61	PI	[29]
18	70	NO	[29]
19	73	PI	[29]
20	79	NO	[29]

Embedded cubature formulae

Degrees	N	Quality	References
$1, 3, \ldots, 2s+1$	$\binom{s+3}{s}$	NI	[44]
2, 3, 4, 5	4, 7, 10, 13	PPPNB	[52]
	4, 7, 10, 16	PPPPB	[52]
	3, 7, 10, 16	PNNNI(2)	[52]
5, 8	7, 19	PPI	[63]
5, 7	7, 16	PNI	[20]
5, 7, 9	7, 16, 28	PNNI	[41]

4. Cubature formulae for three-dimensional regions

4.1. Cubature formulae for the cube C_3

Degree	N	Quality	References
1	1*	PI	[98]
	8	PI(2)	[2]
2	4*	PI	[98]
		PI	[100]
3	6*	PB	[98]
	12	PI	[2]
	14	PI	[2]
	16	PI	[2]
4	10*	PO	[100]
	11	PO	[2]
	13	PI(2)	[2]
5	13*	PI	[98]
		PI	[77]
		PI	[2]
	15	NI	[39]
		PO	[2]
	21	PI	[39]
		PI	[2]
	22	PI(2)	[2]
	26	PI	[39]
	32	EI	[28]
7	27	PO	[98]
	32	PO	[3]
	33	NB	[83]
	35	PB	[36]
9	52	NO	[72]
	53	NO	[72]
		NO	[31]
	56	NI	[72]
	57	NO	[72]
		NI	[72]
		NI	[36]
		NB(2)	[36]
	58	PI	[36]
	59	PB	[36]
		PI	[36]
	62	PI	[36]
	63	PB	[36]
		PI	[36]
11	89	NO	[31]
		NI	[36]
	90	PI(2)	[36]
	91	PI(2)	[36]
		PB(2)	[36]
	96	PI	[36]
		PB	[36]
	97	PI	[36]
		PB	[36]
13	151	NO	[31]
15	223	NO	[3]
	235	NO	[31]
17	307	NO	[31]
19	435	NO	[31]

Embedded cubature formulae

Degrees	N	Quality	References
5, 7	25, 33	NNI	[42]
5, 9	27, 57	NNI	[6]
7, 9	40, 64	(P)NI	[7]
	41, 65 P	?PI	[89]
	41, 65	(N)PI	[7]
	41, 65 P	(N)PI	[7]
	41, 65 P	(P)NI	[7]
	71, 71	(N)PI	[7]
	33, 71	(N)NI	[6]
	45, 77	(N)NI	[6]
	45, 125	(N)PI	[6]
9, 9	77, 125	(N)PI	[6]
9, 11	91, 115	(N)PI	[7]
	71, 137	(N)NI	[6]
	103, 151	(N)NI	[6]
	168, 216	(P)PI	[6]
1, 3, 5, 7, 9, 11, 13	See [43, p.586] with $n = 3$	?I(∞)	[43,22]
1, 3, 5, 7, 9, 11, 13, 15, 17	See [43, p.586] with $n = 3$	?O(∞)	[43,22]

4.2. Cubature formulae for the sphere S_3

Degree	N	Quality	References	Degree	N	Quality	References	Degree	N	Quality	References
4	11	PI	[2]	9	45	PI	[61]			PO	[4,3]
					52	PO(2)	[72]		82	PO	[3]
5	13*	PI	[98]		53	PI	[61]		83	NI	[3]
	14	NI	[48]			PO(3)	[72]		84	PI(2)	[93]
	15	PI	[2]		55	PO	[61]			PI	[93,3]
		PI	[48]		56	PI	[72]		86	PI	[49]
	30	EI	[28]			PI	[93]				
					63	PI	[61]	13	126	NO	[3]
7	27	PO	[98]		71	NI	[61]		127	NI	[3]
	32	PB	[84]		73	PI	[61]		137	PO	[4,3]
		PI	[84]		81	NI	[61]		145	NI	[3]
		PI	[93]								
	33	PI	[27]	11	77	PO	[4]	15	223	NO	[3]

4.3. Cubature formulae for the space $E_3^{r^2}$

Degree	N	Quality	References
5	13*	P	[98]
		P	[77]
	19	P	[85]
7	27	P	[98]
	32	P	[86]
9	45	P	[61]
	52	P	[72]
	53	P	[61]
	55	P	[61]
	63	P	[61]
	71	N	[61]
	81	N	[61]
11	77	P	[4,3]
	86	P	[4,3]
13	127	N	[3]
	137	P	[49]

Embedded cubature formulae

Degrees	N	Quality	References
$1, 3, 5, \ldots, 2m+1$	See [43, p.586] with $n = 3$	?	[22]

4.4. Cubature formulae for the space E_3^r

Degree	N	Quality	References
5	13*	P	[98]
		P	[77]
7	27	P	[98]
	32	P	[86]
9	45	P	[61]
	53	P(2)	[72]
		P	[61]
	55	P	[61]
	63	P	[61]
	71	N	[61]
	81	N	[61]
11	82	P	[3]
	86	P	[4,3]
13	127	N	[3]
	137	P	[4,3]

4.5. Cubature formulae for the tetrahedron T_3

Degree	N	Quality	References
1	1*	PI	[98]
	4	PI	[34]
2	4*	PI	[98]
		PI	[54]
		PB	[54]
		PB	[90]
	5	NB	[25]
	10	NI	[34]
3	5*	NI	[98]
	8	PB	[98]
		PI	[34]
		PI	[54]
		NI	[54]
4	11	NI	[59]
	14	PI	[59]
	15	PB	[54]
		NB	[25]
	16	PI	[57]
		PI	[2]
	29	PB	[1]
	35	NI	[34]
5	14	PI	[44]
		?O	[41]
	15	PI	[59]
		NI	[75]
		NI	[44,25]
	17	PI	[57]
	27	PI	[54]
	56	NI	[34]
6	24	PI	[59]
	29	PI	[57]
	40	PB	[54]
	66	NI	[34]
7	31	NI	[59]
	33	NI(2)	[5,3]
8	43	NI	[5,3]
	45	NI	[59]
9	53	NO	[5,3]

Embedded cubature formulae

Degrees	N	Quality	References
$1, 3, \ldots, 2s+1$	$\binom{s+4}{s}$	NI	[44]

5. Cubature formulae for four-dimensional regions

5.1. Cubature formulae for the four-cube C_4

Degree	N	Quality	References
3	8	EI(2)	[39]
5	24	PI	[95]
	25	NI	[39]
	33	?I	[89]
	41	PI	[39]
	64	EI	[28]

Degree	N	Quality	References
7	49	NO	[98]
		NO	[96]
	57	PI	[71,96]
	73	PI	[27]
9	145	NI	[94]
	153	NI(5)	[94]
	160	PI(5)	[94]
	161	PI(9)	[94]

Embedded cubature formulae

Degrees	N	Quality	References
5, 7	41, 57	NNI	[42]
	33, 65 P	NPI	[95]
	33, 65 P	?I	[89]
7, 9	105, 169	NPI	[95]
3, 5, 7, 9, 11, 13	See [43, p.586] with $n=4$	?I(∞)	[43,22]
3, 5, 7, 9, 11, 13, 15, 17	See [43, p.586] with $n=4$	?O(∞)	[43, 22]

5.2. Cubature formulae for the four-simplex T_4

Degree	N	Quality	References
1	1*	PI	[98]
2	5*	PI	[98]
		PB	[90]
	6	NB	[25]
3	6*	NI	[98]
		NI	[44,25]
4	21	NB	[25]
5	21	NI	[44,25]

6. Cubature formulae for five-dimensional regions

6.1. Cubature formulae for the space $E_5^{r^2}$

Embedded cubature formulae

Degrees	N	Quality	References
$1, 3, 5, \ldots, 2m+1$	See [43, p.586] with $n=5$	?	[22]
$1, 3, 5, 7, 9$	$1, 11, 411, 461, 565$	?	[26]

7. Cubature formulae for n-dimensional regions

Region	Degree	N	Quality	References
C_n $(2 \leqslant n \leqslant 10)$	5	$12 \leqslant N \leqslant 23680$	EI	[28]
C_n	7	$8\binom{n}{3}+4\binom{n}{2}+4n+1$	PI if $n \leqslant 4$ NI if $n \geqslant 5$	[27]
C_n	7	$8\binom{n}{3}+16\binom{n}{2}+4n+1$	NI	[66]
S_n $(2 \leqslant n \leqslant 10)$	5	$12 \leqslant N \leqslant 8000$	EI	[28]
S_n	7	$8\binom{n}{3}+4\binom{n}{2}+4n+1,\ n \neq 5, 6$ $101,\ n=5$ $293,\ n=6$	PI if $n \leqslant 4$ NI if $n \geqslant 5$	[27]
$E_n^{r^2}$	5	$2n^2+1,\ n \neq 4$ $25,\ n=4$	P if $n \leqslant 4$ N if $n \geqslant 5$	[67, 85]
$E_n^{r^2}$ $(4 \leqslant n \leqslant 13)$	5	$72 \leqslant N \leqslant 53952$	E	[28]
$E_n^{r^2}$	7	$8\binom{n}{3}+16\binom{n}{2}+4n+1$	N	[67]
T_n	2	$n+1$*	PI	[90]
T_n	5	$\binom{n+3}{2}$	NI	[75,44]

Embedded cubature formulae

Region	Degrees	N	Quality	References
C_n	5, 7	$(2n^2+2n+1), (2^n+2n^2+2n+1)$	NNI	[42]
C_n	$1, 3, \ldots, 13$	See [43, p.586]	?I	[43,22]
C_n	$1, 3, \ldots, 17$	See [43, p.586]	?O	[43,22]
$E_n^{r^2}$	$1, 3, \ldots, 2m+1$	See [43, p.586]	?	[22]
T_n	$1, 3, \ldots, 2m+1$	$\binom{n+m+1}{m}$	NI	[44]

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